ALM - Mean-Variance Analysis

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Course Program

- Basic interest rate theory
- Interest rate risk management
- Stochastic term structure models
- Risk measurement
- Reinsurance and insurance-linked securities
- `Mean-variance analysis for ALM`

Introduction

- The goal of this section is to show how to hedge a stochastic liability using correlated assets.
- We will also see how to simulate the stochastic development of correlated assets and liabilities.
- Make sure you understand the `matrice operations` and the simulations.

Contents organization

- Optimum asset allocation for one period
  - Minimum risk portfolio
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- Optimum asset allocation to fund a stochastic liability
  - Minimum risk portfolio
  - Optimal portfolio of risky assets
  - Optimal portfolio with a risk-free asset
- Discussion of the mean-variance framework.

MVA: Optimum asset allocation - I

- Assume that you can invest an amount of $W(0)$ in n different assets numbered $1, \cdots, n$.
- The market value of the assets now is $A_1(0), \cdots, A_n(0)$.
- You will revalue your assets at time $t > 0$.
- The market value of the assets will be $A_1(t), \cdots, A_n(t)$. This value must include the value of coupons or dividends paid during the period $(0, t]$.
- To all but the insiders, the outcome of $A_1(t), \cdots, A_n(t)$ looks random.

MVA: Optimum asset allocation - II

- Define the `return` of asset i by

$$R_i(t) = \frac{A_i(t) - A_i(0)}{A_i(t)}.$$

- If you invest $w_i W(0)$ in asset i at time 0, your wealth at time t will be

$$W(t) = W(0) \sum_{i=1}^{n} w_i \left(1 + R_i(t)\right) = W(0)(1 + \mathbf{w}'\mathbf{R}(t)),$$

MVA: Optimum asset allocation - III

- Your aggregate return over the period will be $R_{\mathbf{w}}(t) = \mathbf{w}'\mathbf{R}(t)$.
- The asset allocation problem is to find a vector $\mathbf{w} = (w_1, \cdots, w_n)$ with $w_1 + \cdots + w_n = 1$ that provides an adequate expected return with as little as possible uncertainty.
- You decide to measure the uncertainty of $\mathbf{w}'\mathbf{R}(t)$ by its variance.

MVA: Optimum asset allocation - IV

- In mathematical terms, the `asset allocation problem` then becomes *minimise* $\mathrm{Var}\left(\mathbf{w}'\mathbf{R}(t)\right)$, *subject to certain constraints"*.
- Conceivable investment constraints could be
- `No constraints` at all, i.e. outright minimisation of the variance;
- An `adequate expected return` r, i.e. $\mathrm{E}(\mathbf{w}'\mathbf{R}(t)) = r$;
- `Exposure limits`, e.g. $w_{\min} \leq w \leq w_{\max}$.

MVA: Optimum asset allocation - V

We drop t for now, as we are considering only one period.

- Assume that the `return vector` $\mathbf{R} = (R_1, \cdots, R_n)'$ is random with a known mean vector $\mu = \mathrm{E}(\mathbf{R}) = (\mu_1, \cdots, \mu_n)'$
- and a known `covariance matrix`

$$\Sigma = \mathrm{Cov}(\mathbf{R}) = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \cdots & \sigma_n^2 \end{pmatrix}$$

MVA: Optimum asset allocation - VI

- We assume that there are `only risky assets`: there exists neither an asset i nor a linear combination (portfolio) of assets with a secure return.
- In that case the covariance matrix Σ is `invertible` and `positive definite`.
- A portfolio characterized by the allocation vector $\mathbf{w}$ has expected return and variance as follows:
  $\mathrm{E}(\mathbf{w}'\mathbf{R}) = \mathbf{w}'\mu, \quad \mathrm{Var}(\mathbf{w}'\mathbf{R}) = \mathbf{w}'\Sigma\mathbf{w}.$

Asset behaviour example - assumptions

https://www.global-rates.com/en/interest-rates/central-banks/central-banks.aspx

Minimum Variance Portfolio I

- A very variance-averse investor could pose the asset allocation problem $\min_{\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w}, \text{ subject to (only) } \mathbf{w}'\mathbf{1} = 1.$
- Using Lagrange minimization, the optimal portfolio can be shown to be $\mathbf{w}_{\min} = (\mathbf{1}'\Sigma\mathbf{1})^{-1}\Sigma^{-1}\mathbf{1}$

Minimum Variance Portfolio II

- Its return has expected value and variance as follows:
  $\mu' \mathbf{w}_{\min} = (\mathbf{1}' \Sigma \mathbf{1})^{-1} \mu' \Sigma^{-1} \mathbf{1}; \quad \mathbf{w}'_{\min} \Sigma \mathbf{w}_{\min} = (\mathbf{1}' \Sigma^{-1} \mathbf{1})^{-1}$

Outline of proof

- The Lagrangian can be written as
$$\mathrm{L}(\mathbf{w}, \lambda) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} - \lambda(\mathbf{w}'\mathbf{1} - 1).$$
- To determine $\mathbf{w}_{\min}$ we solve the linear equations
$$\frac{\partial}{\partial \mathbf{w}}\mathrm{L}(\mathbf{w}, \lambda) = \mathbf{w}'\Sigma - \lambda\mathbf{1}' = 0',$$
$$\frac{\partial}{\partial \lambda}\mathrm{L}(\mathbf{w}, \lambda) = \mathbf{w}'\mathbf{1} - 1 = 0.$$

Example Min. Variance Portfolio

Optimal portfolio of risky assets - I

- A more demanding investor could pose the asset allocation problem
  $\min_{\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w}, \text{ subject to } \mathbf{w}'\mu = \mathrm{r} \text{ and } (\text{of course}) \ \mathbf{w}'\mathbf{1} = 1,$
  where r is the `expected return that an allocation must provide` in order to be a candidate.
- The optimal portfolio $\mathbf{w}_r$ is now a `linear combination` of the minimum variance portfolio $\mathbf{w}_{\min}$ and one "reference" risky portfolio $\mathbf{w}_{\text{ref}}$:
  $\mathbf{w}_r = (1-\nu)\mathbf{w}_{\min} + \nu\mathbf{w}_{\text{ref}}$

Optimal portfolio of risky assets - II

- The `reference risky portfolio` is
  $\mathbf{w}_{\text{ref}} = \left(\mathbf{1}'\Sigma^{-1}\mu\right)^{-1}\Sigma^{-1}\mu$
- or, in special cases, $\mathbf{w}_{\text{ref}} = \mathbf{w}_{\min} + \Sigma^{-1}\mu$.
- The `weight` of the risky portfolio in the optimal portfolio is
  $\nu = \nu(r) = \dfrac{r - \mu'\mathbf{w}_{\min}}{\mu'\mathbf{w}_{\mathbf{ref}} - \mu'\mathbf{w}_{\min}}.$
- The more return you ask for, the more risk you must accept.

Outline of proof - I

- The Lagrangian can be written as

$$\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_2(\mathbf{w}'\mu - r).$$

- To determine $\mathbf{w}_r$ we solve the linear equations

1. $\frac{\partial}{\partial \mathbf{w}}\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\Sigma - \lambda_1\mathbf{1}' - \lambda_2\mu' = \mathbf{0}'$
2. $\frac{\partial}{\partial \lambda_1}\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mathbf{1} - 1 = 0$
3. $\frac{\partial}{\partial \lambda_2}\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mu - r = 0$

Outline of proof - II

- Using 1 we find that the solution $\mathbf{w}$ is of the form
  $\mathbf{w} = \lambda_1 \Sigma^{-1}\mathbf{1} + \lambda_2 \Sigma^{-1}\mu = \lambda_1(\mathbf{1}'\Sigma^{-1}\mathbf{1})\mathbf{w}_{\min} + \lambda_2 \Sigma^{-1}\mu.$
- Inserting this into 2 we find
  $\lambda_1(\mathbf{1}'\Sigma^{-1}\mathbf{1}) = 1 - \lambda_2(\mathbf{1}\Sigma^{-1}\mu).$
- If $\mathbf{1}\Sigma^{-1}\mu \neq 0$, we can write
  $\mathbf{w} = (1-\nu)\mathbf{w}_{\min} + \nu\mathbf{w}_{\text{ref}},$

Outline of proof - III

with a reference portfolio that is
$\mathbf{w} = (\mathbf{1}'\Sigma^{-1}\mu)^{-1}\Sigma^{-1}\mu.$

- If $\mathbf{1}\Sigma^{-1}\mu = 0$, we can still write
  $\mathbf{w} = (1-\nu)\mathbf{w}_{\min} + \nu\mathbf{w}_{\text{ref}},$
  but the reference portfolio becomes (proof as an exercise)
  $\mathbf{w}_{\text{ref}} = \mathbf{w}_{\min} + \Sigma^{-1}\mu.$
- Finally, we solve 3 to determine the weight of the reference portfolio:
  $$\nu = \nu(r) = \frac{r - \mu'\mathbf{w}_{\min}}{\mu'\mathbf{w}_{\text{ref}} - \mu'\mathbf{w}_{\min}}$$

Example Reference Risky Portfolio

Example - Optimal portfolio of risky assets with return requirement

The efficient frontier of risky assets

- Any required (expected) return r can be generated by the formula
  $\mathbf{w}_r = (1-\nu(r))\mathbf{w}_{\min} + \nu(r)\mathbf{w}_{\text{ref}}$,
- and the variance of the return will be the least possible:

$$\begin{aligned}\sigma^2(r) &= \mathbf{Var}(\mathbf{w}_r'\mathbf{R}) \\ &= (1-\nu(r))^2\mathbf{w}_{\min}'\Sigma\mathbf{w}_{\min} + \nu^2(r)\mathbf{w}_{\text{ref}}'\Sigma\mathbf{w}_{\text{ref}} \\ &+ 2(1-\nu(r))\nu(r)\mathbf{w}_{\min}'\Sigma\mathbf{w}_{\text{ref}}.\end{aligned}$$

The efficient frontier - II

- The `efficient frontier of risky assets` is the curve
  $\{(\sigma(r), r) : r \geq \mu' \mathbf{w}_{\min}\}$
  in the two-dimensional plane (proof as an exercise).

Optimal portfolio with a risk-free asset - I

- Assume now that in addition to the n risky assets, you can invest in a `risk-free asset` ($i = 0$) that provides a `secure return` of $R_0 = \mu_0$.
- Your asset allocation problem now becomes
$$\min_{w_0, \mathbf{w}} \mathbf{w}'\Sigma\mathbf{w}, \quad \text{subject to} \quad w_0\mu_0 + \mathbf{w}'\mu = r \quad \text{and} \quad w_0 + \mathbf{w}'\mathbf{1} = 1,$$
- where r is the expected return that an allocation must provide in order to be a candidate,
- and w_0 is the proportion of your wealth to be invested risk-free.

Optimal portfolio with a risk-free asset - II

- In this case, the optimal portfolio is a `combination` of
- a risk-free investment of w_0 and
- investment of the remaining $1 - w_0$ in a tangency portfolio $\mathbf{w}_{\tan}$.

Optimal portfolio with a risk-free asset - III

- The relevant parameters are

$$\mathbf{w}_{\tan} = \mathbf{w}_{\tan}(\mu_0) = \left(\mathbf{1}'\Sigma^{-1}(\mu - \mu_0\mathbf{1})\right)^{-1}\Sigma^{-1}(\mu - \mu_0\mathbf{1})$$

$$1 - w_0 = 1 - w_0(r) = \frac{r - \mu_0}{\mu'\mathbf{w}_{\tan} - \mu_0}$$

Outline of proof I

- The Lagrangian can be written as

$$\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} - \lambda_1(w_0 + \mathbf{w}'\mathbf{1} - 1) - \lambda_2(w_0\mu_0 + \mathbf{w}'\mu - r).$$

- To determine the optimal $(w_0, \mathbf{w})$ we solve the linear equations

4. $\frac{\partial}{\partial \mathbf{w}}\mathrm{L}(w_0, \mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\Sigma - \lambda_1\mathbf{1}' - \lambda_2\mu' = \mathbf{0}'$
5. $\frac{\partial}{\partial \mathbf{w}_0}\mathrm{L}(w_0, \mathbf{w}, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2\mu_0 = 0$
6. $\frac{\partial}{\partial \lambda_1}\mathrm{L}(w_0, \mathbf{w}, \lambda_1, \lambda_2) = w_0 + \mathbf{w}'\mathbf{1} - 1 = 0$
7. $\frac{\partial}{\partial \lambda_2}\mathrm{L}(w_0, \mathbf{w}, \lambda_1, \lambda_2) = w_0\mu_0 + \mathbf{w}'\mu - r = 0$

Outline of proof II

- Using 4. we find that the solution $\mathbf{w}$ is of the form $\mathbf{w} = \lambda_1 \Sigma^{-1} \mathbf{1} + \lambda_2 \Sigma^{-1} \mu$.
- Using 5. we find that $\lambda_1 = -\lambda_2 \mu_0$, so that $\mathbf{w} = \lambda_2 \Sigma^{-} 1 (\mu - \mu_0 \mathbf{1})$
- Using 6. we find $\lambda_2 = \frac{1 - w_0}{\mathbf{1}' \Sigma^{-1} (\mu - \mu_0 \mathbf{1})}$, so that $\mathbf{w} = (1 - w_0) \mathbf{w_{tan}}$.

Outline of proof III

- Finally, 7. gives us
  $$1 - w_0 = 1 - w_0(r) = \frac{r - \mu_0}{\mu' \mathbf{w}\mathrm{tan} - \mu_0}.$$
- Note that the tangency portfolio is a function of the available risk-free return.
- The variance of the overall return is
  $\sigma^2(r) = \mathrm{Var}(w_0\mu_0 + (1 - w_0)\mathbf{w}'_{\mathrm{tan}}\mathrm{R}) = (1 - w_0)^2\mathbf{w}'_{\mathrm{tan}}\Sigma\mathbf{w}_{\mathrm{tan}}.$

Example - Tangency Portfolio

Example

- Optimal Portfolio of risky assets and a risk-free asset with return requirement

Optimum asset allocation with hedge

- Let us briefly reintroduce the time parameter $t > 0$.
- Assume that the assets must support a `stochastic liability`.
- The value of the liability at time 0 is $\mathrm{L}(0)$, and the time t it will be $\mathrm{L}(t)$.
- The surplus at time 0 is $\mathrm{S}(0) = \mathrm{W}(0) - \mathrm{L}(0)$. At time t it will be $\mathrm{S}(t) = \mathrm{W}(t) - \mathrm{L}(t)$.
- The funding ratio at time 0 is $\mathrm{F}(0) = \frac{\mathrm{W}(0)}{\mathrm{L}(0)}$.

Optimum asset allocation with hedge

- Sharpe & Tint (1990) define the `surplus return` as

$$\frac{\mathrm{S}(t)-\mathrm{S}(0)}{\mathrm{W}(0)} = \left(\frac{\mathrm{W}(t)-\mathrm{W}(0)}{\mathrm{W}(0)}\right) - \frac{\mathrm{L}(0)}{\mathrm{W}(0)}\left(\frac{\mathrm{L}(t)-\mathrm{L}(0)}{\mathrm{L}(0)}\right)$$
$$= \mathrm{R}_{\mathrm{W}}(t) - \frac{\mathrm{R}_{\mathrm{L}}}{\mathrm{F}(0)}.$$

- Here we have defined:
- $\mathrm{R}_{\mathrm{W}}(t) = \frac{\mathrm{W}(t)-\mathrm{W}(0)}{\mathrm{W}(0)}$ as asset return
- $\mathrm{R}_{\mathrm{L}}(t) = \frac{\mathrm{L}(t)-\mathrm{L}(0)}{\mathrm{L}(0)}$ as liability growth.

Assumptions - I

- Let us assume that there are n investible assets with a random return characterized by its mean vector and covariance matrix:
  $\mathbf{R} \sim [\mu(t), \Sigma(t)]$
- We now make the `additional assumption` that liability growth is random and correlated with asset returns:

$$\begin{aligned} \mathrm{E}(\mathrm{R_L}(t)) &= \mu_{\mathrm{L}}(t) \\ \mathrm{Var}(\mathrm{R_L}(t)) &= \sigma^2_{\mathrm{L}}(t) \\ \mathrm{Cov}(\mathrm{R}_i(t), \mathrm{R_L}(t)) &= \gamma_{i,\mathrm{L}}(t) = \rho_{i,\mathrm{L}}(t)\sigma_i(t)\sigma_{\mathrm{L}}(t) \end{aligned}$$

Assumptions - II

- Denote the vector of covariances by
  $\gamma(t) = (\gamma_{1,\mathrm{L}}(t), \cdots, \gamma_{n,\mathrm{L}}(t))'$
  and assume you know (have estimated) $\mu_{\mathrm{L}}(t)$, $\sigma^2_{\mathrm{L}}(t)$ and $\gamma(t)$.
- Let us now find optimal asset allocations to fund a stochastic liability.

Optimum asset allocation with hedge

- With an arbitrary asset allocation vector $\mathbf{w}$, the random surplus return is

$$\mathrm{R_S}(t) = \mathbf{w}'\mathbf{R}(t) - \frac{\mathrm{R_L}(t)}{\mathrm{F}(0)} = \mathbf{w}'\mathbf{R} - \frac{\mathrm{R_L}}{\mathrm{F}} = \mathrm{R_S}$$

- It is easy to verify that

$$\mathrm{E}(\mathrm{R_S}) = \mathbf{w}'\mu - \frac{\mu_\mathrm{L}}{\mathrm{F}}$$

$$\mathrm{Var}(\mathrm{R_S}) = \mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_\mathrm{L}^2}{\mathrm{F}^2} - 2\frac{\mathbf{w}'\gamma}{\mathrm{F}}$$

- Let us minimize the variance, subject to the constraints.

Minimum Variance Portfolio - Hedge

- If your only aim is to `minimize variance`, you would solve:
  $$\min_{\mathbf{w}} \left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_{\mathrm{L}}^2}{\mathrm{F}^2} - 2\frac{\mathbf{w}'\gamma}{\mathrm{F}} \right) \text{ subject to } \mathbf{w}'\mathbf{1} = 1$$
- Using Lagrange minimization, the optimal portfolio can be shown to be
  $\mathbf{w}_{\min}(\mathrm{F}, \gamma) = (1 - \nu)\mathbf{w}_{\min} + \nu\mathbf{w}_{\gamma}$
  where $\mathbf{w}_{\min}$ is the unconditional minimum variance portfolio and $\mathbf{w}_{\gamma}$ is the `liability hedge portfolio`.

Liability Hedge Portfolio

- The liability hedge portfolio is
  $\mathbf{w}_\gamma = (\mathbf{1}'\Sigma^{-1}\gamma)^{-1}\Sigma^{-1}\gamma$.
- The weight of the liability hedge portfolio in the optimal portfolio is
  $\nu = \nu(\mathrm{F}, \gamma) = \dfrac{1}{\mathrm{F}}\mathbf{1}'\Sigma^{-1}\gamma$.
- In the case where $\mathbf{1}'\Sigma^{-1}\gamma = 0$, we can write $\mathbf{w}_\gamma = \mathbf{w}_{\min} + \Sigma^{-1}\gamma$ and $\nu = 1$.

Outline of proof - I

- The Lagrangian can be written as

$$\mathrm{L}(\mathbf{w}, \lambda) = \frac{1}{2}\left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_{\mathrm{L}}^2}{\mathrm{F}^2} - 2\frac{\mathbf{w}'\gamma}{\mathrm{F}}\right) - \lambda(\mathbf{w}'\mathbf{1} - 1).$$

- To determine $\mathbf{w}$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}}\mathrm{L}(\mathbf{w}, \lambda) = \mathbf{w}'\Sigma - \frac{1}{\mathrm{F}}\gamma' - \lambda\mathbf{1}' = \mathbf{0}',$$

$$\frac{\partial}{\partial \lambda}\mathrm{L}(\mathbf{w}, \lambda) = \mathbf{w}'\mathbf{1} - 1 = 0$$

Outline of proof - II

- The first equation gives

  $$\mathbf{w} = \lambda \Sigma^{-1} \mathbf{1} + \frac{1}{\mathrm{F}} \Sigma^{-1} \gamma'$$

- and the second equation gives

  $$\lambda = (\mathbf{1}`\Sigma^{-1}\mathbf{1})-1 \left(1 - \frac{1}{\mathrm{F}} \mathbf{1}' \Sigma^{-1} \gamma\right).$$

- Proceed from there.

Comments

- If asset-liability `covariance is small` relative to the asset variability, then ν will be small and the optimal portfolio will be close to $\mathbf{w}_{\min}$.
- In particular, if there is no asset-liability covariance then the optimal portfolio is just $\mathbf{w}_{\min}$.
- The weight given to the liability hedge portfolio is a `decreasing` function of the initial funding ratio.

Example - Liability hedge portfolio

Example - Minimum variance portfolio with hedge

Optimal portfolio of risky assets with hedge I

- If one wants to beat instead of mee the expected return of the liability hedge portfolio, one could solve:

  $$\min_{\mathbf{w}} \left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_{\mathrm{L}}^2}{\mathrm{F}^2} - 2\frac{\mathbf{w}'\gamma}{\mathrm{F}} \right) \text{ subject to } \mathbf{w}'\mu = r \text{ and } \mathbf{w}'\mathbf{1} = 1$$

  where r is the expected return required.
- This only makes sense if $r \geq \mu'\mathbf{w}_{\min}(\mathrm{F}, \gamma)$.

Portfolio of risky assets with hedge II

- The optimal portfolio can be written as:

$$\begin{aligned}\mathbf{w}_r(\mathrm{F},\gamma) &= (1-\nu-\omega)\mathbf{w}_{\min} + \omega\mathbf{w}_{\text{ref}} + \nu\mathbf{w}_\gamma \\ &= \mathbf{w}_{\min}(\mathrm{F},\gamma) + \omega(\mathbf{w}_{\text{ref}} - \mathbf{w}_{\min})\end{aligned}$$

- $\mathbf{w}_{\min}$ is the unconditional minimum variance allocation,
- $\mathbf{w}_{\text{ref}}$ is the reference risky portfolio without a risk-free asset,
- $\mathbf{w}_\gamma$ is the `liability hedge` portfolio,
- $\mathbf{w}_{\min}(\mathrm{F},\gamma)$ is the `minimum surplus variance` allocation.
- The weighting parameters are

$$\nu = \frac{1}{\mathrm{F}}\mathbf{1}'\Sigma^{-1}\gamma \text{ and } \omega = \frac{r - \mu'\mathbf{w}_{\min}(\mathrm{F},\gamma)}{\mu'\mathbf{w}_{\text{ref}} - \mu'\mathbf{w}_{\min}}.$$

Outline of proof

- The Lagrangian can be written as

$$\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_{\mathrm{L}}^2}{\mathrm{F}^2} - 2\frac{\mathbf{w}'\gamma}{\mathrm{F}}\right) - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_2(\mathbf{w}'\mu - r).$$

- To determine $\mathbf{w}$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}}\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\Sigma - \frac{1}{\mathrm{F}}\gamma' - \lambda_1\mathbf{1}' - \lambda_2\mu' = \mathbf{0}',$$

$$\frac{\partial}{\partial \lambda_1}\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mathbf{1} - 1 = 0$$

$$\frac{\partial}{\partial \lambda_2}\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mu - r = 0$$

and so on…

Example

- Optimal portfolio of risky assets to fund a stochastic liability

Portfolio of risky assets with a risk-free asset with hedge - I

- Finally, let us develop the case where the investor has access to a `risk-free asset` with return μ_0, i.e.,

$$\min_{\mathbf{w}} \left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_{\mathrm{L}}^2}{\mathrm{F}^2} - 2\frac{\mathbf{w}'\gamma}{\mathrm{F}} \right) \text{ subject to } \begin{aligned} w_0\mu_0 + \mathbf{w}'\mu &= r \\ \text{and } \mathbf{w}_0 + \mathbf{w}'\mathbf{1} &= 1 \end{aligned}$$

- The parameter w_0 denotes the proportion of assets invested risk-free.

Portfolio of risky assets with a risk-free asset with hedge - II

- The optimal portfolio consists of
- a risk-free investment of w_0,
- investment of $1 - w_0 - \nu$ in portfolio $\mathbf{w_{tan}}$,
- investment of ν in the liability hedge portfolio $\mathbf{w}_\gamma$.
- The weights are

$$\nu = \frac{1}{\mathrm{F}} \mathbf{1}' \Sigma^{-1} \gamma \text{ and } 1 - w_0 = \frac{r - \nu \mu' (\mathbf{w}_\gamma - \mathbf{w}_{\tan}) - \mu - 0}{\mu' \mathbf{w} \tan - \mu_0}$$

- We omit the proof.

Example

- Optimal portfolio of risky assets with a risk-free asset to fund a stochastic liability

Discussion of mean-variance framework - I

- Using the mean-variance framework provides insight into the effect of `correlation` between asset classes, and between asset classes and liabilities.
- Estimating the covariances is easy in principle. Having to rely on estimated covariances in allocating your portfolio may be more problematic. It requires great confidence in the estimates.
- The mean-variance framework may return allocations that are `not feasible`, because they are outside the investment mandate. There exists software to minimize with arbitrary constraints.

Discussion of mean-variance framework - II

- Even if your asset allocation is subject to constraints, you should calculate the `cost` of those constraints in terms of lost return or increased volatility, relative to what an unconstrained allocation could achieve.
- Given the framework (`means and covariances`), the method returns an optimal asset allocation. *Optimal* does not necessarily mean *very good* - it just means the best that could be achieved under the given assumptions.
- Asset returns are `not normally distributed`! However, relying on means and covariances does not imply that you subscribe to the normality assumption. It only means that you select two readily available distribution characteristics and ignore the rest.

Speaker notes